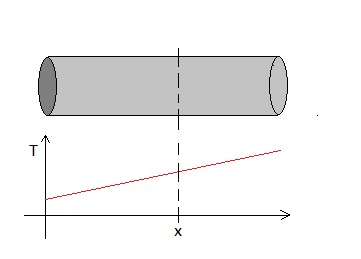
**Thermal Conductivity**

Now let’s look at the thermal conductivity of a metal, coming not from electrons, but phonons. Physically, when the lattice is oscillating, due to temperature, the oscillations generate waves – distortions in the lattice that travel down the length of the metal. These oscillations carry energy and momentum. When we have a thermal gradient, the higher temperature part of the metal will be sending out more energetic waves in both directions, than the lower temperature part. And so since we have an unequal ‘radiance’ of waves from the two temperature locals, we will get net energy flux towards the lower temperature region. We’d like to work out the thermal current as a function of the thermal gradient. So consider our metal with a temperature gradient from one end to the other.



Seems that we’re going to approximate jq as what we’d call, from thermodynamics, jε rather.



where ε is the phonon energy density (energy per unit volume), and v is the velocity of the phonons (note that the requirement that our phonons have appreciable velocity rules out appreciable thermal conductivity from unscreened phonons). For weak temperature gradients, it will follow:



and we’d like to extract this coefficient κ. We will assume that the temperature of the metal increases from left to right – hence the gradient – and consequently there will be a phonon flux across the line at x, from right to left (because phonons on right are traveling faster than those on left). The phonons at the arbitrary position x will have an equilibrium energy density ε(T(x)). At this position x, some phonons will be flowing to the right, and some to the left. We would assume that half would be going to the right and half to the left. The velocity of those going to the right would be vs, and the velocity of those going to the left would also be vs, since the speed of phonons is constant, at least according to the Debye model. The temperature of those phonons depends on where they last collided. We can say that the right going phonons will have temperature T(x-vst) and the left going ones T(x+vst), where t is the time to their last collision. t is a random variable, ranging from 0 to ∞. But its average is τ. And if we just replace all t’s by the average t, the net current across the threshold will be,



Note we’re presuming that collisions between phonons is what supports thermal equilibration. So technically, we’re assuming that there is a phonon interaction present. This would come about from anharmonic terms in the Hamiltonian (basically what we’d get if we expanded the Hamiltonian out to greater than 2nd order in the ion displacements).

A diagram of a complex geometry

Description automatically generated with medium confidence

So collisions would, via Umklapp processes, cause phonons to rather randomly get scattered into positive or negative velocity states. This is illustrated above, where two rightward going orange phonons could collide, and end up both reversing direction (at least, that’s one possible outcome). Since the details of the interaction are not important for us here, I’m placing the analysis in this file, despite the fact that we’re supposed to be dealing with ‘free’ phonons. Expanding for small τ, and first order in vs,



where cV is the heat capacity per unit volume. Now should make a slight change. The vs2 we have is the squared velocity in the x-direction. If we let vs2 be the overall velocity squared, then the squared velocity in the x-direction would be vs2/3 (since vx2 = vy2 = vz2). And so we have (seems like we’ve calculated the energy current, jε, rather than the heat current per se´),



We’ll observe that if τ → ∞, i.e., if collisions were non-existent, the thermal conductivity would likewise go to ∞. This is follows from the fact that the left going phonons across point x would have temperature T(x+vsτ) → ∞, and the right going ones would have temperature T(x-vsτ) → 0. So it is these Umklapp processes that make the thermal conductivity finite. Let’s work this coefficient out a little. First, vs is a constant, independent of temperature. And for T > TE (which is around 100K, so room temperature), we can take the heat capacity to be:



Further, I think we’d take the scattering rate 1/τ as proportional to the number of phonons present. For T > TE, we should have:



and we can approximate Ωks ~ ΩE, as the branches are pretty flat. So I think we’d have:



And so we should expect,



Near room temperature, this simplifies to:



So interestingly, it would decrease with increasing temperature, because more phonons would impede transport. Turns out this thermal conductivity is exceedingly small, however. Because the energy branches are practically flat, vks = ∂ωks/∂k is practically zero. We’ll find we need to incorporate the electron-phonon interaction to get appreciable thermal conductivity, as this will basically pull the small k part of the spectrum into the origin.

**2nd Sound**

While we’re on the topic of phonon collisions. Apparently, just as ions can collide, and transfer energy/momentum to one another, phonons can do the same. And just as momentum/energy can be transferred in the form of sound waves for lattice ions, momentum/energy can be transferred in the form of (2nd) sound waves for phonons.